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THE RATE OF GROWTH OF SAMPLE MAXIMA

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The rate of growth of sample maxima*)

by

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Introduction. Suppose X_1, X_2, X_3, ... are independent real-valued random variables with common distribution function F. Suppose F has a positive derivative F'(x) for all sufficiently large x. We define

$$Y_n = \max (X_1, X_2, \dots, X_n).$$

From Von Mises' work [4] we know that weak convergence properties of $\{Y_n\}$ are closely related to the behaviour of the function f defined by

$$(1) \qquad f(x) = \frac{1-F(x)}{F'(x)}$$

for $x \to \infty$. It will be shown that much about the sample behaviour of $\{Y_n\}$ can be concluded from the behaviour of the function g defined by

$$(2) \qquad g(x) = \frac{\{1-F(x)\} \log \log\{1/1-F(x)\}}{F'(x)}$$

for $x \to \infty$.

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*) Report SW 6/71 of the Department of Mathematical Statistics of the Mathematical Centre, Amsterdam.

Our exposition is based on a few lemmas of an analytic nature which are proved in section 1. In section 2 first we give conditions under which almost surely

$$0 < \liminf_{n\to\infty} Y_n/b_n \leq \limsup_{n\to\infty} Y_n/b_n < \infty$$

with b_n defined by $F(b_n) = 1 - 1/n$. For the special case that $\lim_{n\to\infty} Y_n/b_n$ exists almost surely, a more refined result is proved which previously is stated by J. Pickands III [5]. However the proof given there seems to contain an error.

Most of our conditions imply that

$$\lim_{n\to\infty} P\{\frac{Y_n-b_n}{f(b_n)} \leq x\} = \exp(-e^{-x}).$$

In section 3 we give a large deviations result in connection with this weak convergence property.

<u>Section 1</u>. In this section we give some lemmas which we need afterwards. The lemmas 1 and 3 play a basic role in our attack.

<u>Lemma 1</u>. Suppose ψ is a real-valued function with positive derivative ψ' and $\lim_{x\to\infty} \psi(x) = \infty$. If for some constant c $(0 \leq c \leq \infty)$

$$(3) \qquad \lim_{t\to\infty} \frac{\log \psi(t)}{t.\psi'(t)} = c,$$

then for all positive x

$$(4) \qquad \lim_{t\to\infty} \frac{\psi(tx)-\psi(t)}{\log \psi(t)} = \frac{\log x}{c} \ .$$

<u>Proof</u>. First suppose $0 < c \leq \infty$. Without loss of generality we assume $\psi(1) = 2$. Define the function p by

$$p(t) = \frac{t.\psi'(t)}{\log \psi(t)} \ ,$$

then

$$\int_1^t \frac{p(s)}{s}\, ds = \int_1^t \frac{\psi'(s)ds}{\log \psi(s)} = \int_2^{\psi(t)} \frac{ds}{\log s}.$$

If we denote the function $\int_2^x \frac{ds}{\log s}$ by I(x) and its inverse function by K, we get

$$(5) \qquad \psi(t) = K\left(\int_1^t \frac{p(s)}{s}\, ds\right).$$

Applying de l'Hospital's rule one sees that

$$\log I(y) \sim \log y \qquad \text{for } y \to \infty.$$

Substitution of x for I(y) gives

$$\log K(x) \sim \log x \qquad \text{for } x \to \infty.$$

Hence

$$(6) \qquad K'(x) = \log K(x) \sim \log x \qquad \text{for } x \to \infty.$$

We now calculate the limit (4). Using (5) we have

$$\frac{\psi(tx)-\psi(t)}{\log \psi(t)} = \frac{K\left(\int_1^{tx} \frac{p(s)}{s}\, ds\right) - K\left(\int_1^t \frac{p(s)}{s}\, ds\right)}{\log \psi(t)} =$$

$$= \frac{K\left(\int_1^x \frac{p(ts)}{s} ds + \int_1^t \frac{p(s)}{s}\, ds\right) - K\left(\int_1^t \frac{p(s)}{s}\, ds\right)}{\log K\left(\int_1^t \frac{p(s)}{s}\, ds\right)} .$$

Consequently

$$\lim_{t\to\infty} \frac{\psi(tx)-\psi(t)}{\log \psi(t)} = \lim_{y\to\infty} \frac{K(y+a(y))-K(y)}{\log K(y)} ,$$

where

$$\lim_{y\to\infty} a(y) = \lim_{t\to\infty} \int_1^x \frac{p(ts)}{s}\, ds = \frac{\log x}{c} .$$

By the mean value theorem of differential calculus we get for some $0 \leq \theta(y) \leq 1$

$$\lim_{t\to\infty} \frac{\psi(tx)-\psi(t)}{\log \psi(t)} = \lim_{y\to\infty} \quad a(y) \quad \frac{K'(y+\theta(y).a(y))}{\log K(y)} =$$

$$= \lim_{y\to\infty} a(y) \frac{\log K(y+\theta(y).a(y))}{\log K\ (y)} = \lim_{y\to\infty} a(y) \frac{\log(y+\theta(y).a(y))}{\log(y)} = \frac{\log x}{c} .$$

For c = 0, the same procedure shows (4) for x > 1. Suppose (4) does not hold for x < 1. Then for some $x_0 > 1$ and sequence $t_n \to \infty$ we have

$$\limsup_{n\to\infty} \frac{\psi(t_n x_0)-\psi(t_n)}{\log \psi(t_n x_0)} < \infty.$$

On the other hand

$$\lim_{n\to\infty} \frac{\psi(t_n x_0)-\psi(t_n)}{\log \psi(t_n)} = \infty,$$

hence

$$\lim_{n\to\infty} \frac{\log \psi(t_n x_0)}{\log \psi(t_n)} = \infty.$$

As clearly for $0 < \xi < \eta$

$$\xi(\log \eta - \log \xi) < \eta - \xi,$$

we have

$$\frac{\psi(t_n)\{\log \psi(t_n x_0)-\log \psi(t_n)\}}{\log \psi(t_n x_0)} < \frac{\psi(t_n x_0)-\psi(t_n)}{\log \psi(t_n x_0)} .$$

As for $n \to \infty$ the lefthand member tends to infinity and the righthand member is bounded, by contradiction we have (4) for all positive x. □

Remark. With the aid of theorem 1.4.2 from section 1.4 of [3] one can prove that for non-decreasing ψ with $\lim_{x\to\infty} \psi(x) = \infty$ and $0 < c < \infty$ relation (4) is equivalent to

$$\lim_{x\to\infty} \frac{\psi(x) - \frac{1}{x}\int_0^x \psi(t)dt}{\log \psi(x)} = \frac{1}{c} .$$

Lemma 2. Suppose f is a positive differentiable function and

$$\lim_{t\to\infty} f'(t) = 0.$$

Then

$$\lim_{t\to\infty} \frac{f(t)}{f(t+x\ f(t))} = 1$$

uniformly on each bounded x-interval.

Proof. By the mean value theorem of differential calculus for some $0 \leq \theta(t,x) \leq 1$

$$f(t+xf(t)) = f(t) + x\ f(t)\ f'(t+\theta(t,x)x\ f(t)).$$

From $f'(t) \to 0$ for $t \to \infty$ we get $t^{-1}f(t) \to 0$ and hence $t + \theta(t,x)x\ f(t) \to \infty$ for all x.Now the statement of the lemma follows as

$$\lim_{t\to\infty} f'(t+\theta(t,x)x\ f(t)) = 0$$

uniformly on each bounded x-interval. □

Lemma 3. Suppose ψ is a twice differentiable real-valued function with positive derivative ψ' and $\lim_{x\to\infty} \psi(x) = \infty$. Define the funtion q by

(7) $$q(t) = \frac{\log \psi(t)}{\psi'(t)}$$

and suppose

$$\lim_{t\to\infty} q'(t) = 0,$$

then for all real x

(8) $$\lim_{t\to\infty} \frac{\psi(t+x.q(t)-\psi(t))}{\log \psi(t)} = x.$$

<u>Proof</u>. We proceed in the same way as in the proof of lemma 1. Again we suppose $\psi(1) = 2$ and get

$$\psi(t) = K(\int_1^t \frac{ds}{q(s)}).$$

Now

$$\frac{\psi(t+x\ q(t))-\psi(t)}{\log \psi(t)} = \frac{K(\int_t^{t+xq(t)} \frac{ds}{q(s)} + \int_1^t \frac{ds}{q(s)}) - K(\int_1^t \frac{ds}{q(s)})}{\log K(\int_1^t \frac{ds}{q(s)})}$$

Consequently

$$\lim_{t\to\infty} \frac{\psi(t+x\ q(t))-\psi(t)}{\log \psi(t)} = \lim_{y\to\infty} \frac{K(b(y)+y)-K(y)}{\log K(y)}$$

where by lemma 2

$$\lim_{y\to\infty} b(y) = \lim_{t\to\infty} \int_t^{t+xq(t)} \frac{ds}{q(s)} = \lim_{t\to\infty} \int_0^x \frac{q(t)}{q(t+s\ q(t))}\, ds = x.$$

In the same way as in the proof of lemma 1 the statement (8) follows. □

The following lemma is of a probabilistic character. The elements for this lemma can be found in [1], [2] and [5]. We consider the situation described in the introduction.

Lemma 4. Suppose $\{c_n\}$ is a sequence of positive constants, $b_n = \inf\{x \mid 1-F(x) \leq 1/n\}$ and $\{c_n x+b_n\}$ is an ultimately non-decreasing sequence for all real $x > -1$.

a) For all distribution functions F we have almost surely

$$\liminf_{n\to\infty} \frac{Y_n-b_n}{c_n} \leq 0.$$

b) Suppose c is a finite constant. We have almost surely

$$\limsup_{n\to\infty} \frac{Y_n-b_n}{c_n} = c$$

if and only if

(9) $$\sum_{n=1}^{\infty} \{1-F(c_n x+b_n)\}$$

converges for all $x > c$ and diverges for all $x < c$.

c) If for all $-1 < x < 0$

(10) $$\sum_{n=1}^{\infty} \{1-F(c_n x+b_n)\} \exp\{-n(1-F(c_n x+b_n))\} < \infty,$$

then almost surely

(11) $$\liminf_{n\to\infty} \frac{Y_n-b_n}{c_n} \geq 0.$$

Proof.

a)

$$P\{Y_n \leq b_n \text{ infinitely often}\} \geq \limsup_{n\to\infty} P\{Y_n \leq b_n\} = \limsup_{n\to\infty} F^n(b_n) \geq$$

$$\geq (1-1/n)^n = e^{-1} > 0.$$

As $\{Y_n \leq b_n$ infinitely often$\}$ is a tail event, we have

$$P\{Y_n/c_n \leq b_n/c_n \text{ infinitely often}\} = P\{Y_n \leq b_n \text{ infinitely often}\} = 1.$$

b) As $\{c_n x+b_n\}$ is a non-decreasing sequence for all real $x > -1$, we have $Y_n > c_n x + b_n$ infinitely often if and only if $X_n > c_n x + b_n$ infinitely often. As the X_n are independent, part b) is a direct consequence of the Borel-Cantelli lemmas.

c) As $\sum_{n=1}^{\infty} \{1-F(b_n)\} = \infty$, we have almost surely $Y_n > b_n$ i.o. Hence also $Y_n > c_n x + b_n$ i.o. for all $x < 0$. So for proving (11) it is sufficient to show that almost surely

$$P\{Y_n \leq c_n + b_n \text{ and } Y_{n+1} > c_{n+1}x + b_{n+1} \text{ finitely often}\} = 1.$$

or equivalently (as $\{c_n x+b_n\}$ is non-decreasing for $x > -1$)

$$P\{Y_n \leq c_n x + b_n \text{ and } X_{n+1} > c_{n+1}x + b_{n+1} \text{ finitely often}\} = 1.$$

By the first Borel-Cantelli lemma this is true if

$$(12) \quad \sum_{n=1}^{\infty} P\{Y_n \leq c_n x + b_n \text{ and } X_{n+1} > c_{n+1}x + b_{n+1}\} =$$

$$= \sum_{n=1}^{\infty} \{1-F(c_{n+1}x+b_{n+1})\}. F^n(c_n x+b_n)$$

converges. Now

$$1 - F(c_{n+1}x+b_{n+1}) \leq 1 - F(c_n x+b_n)$$

and

$$F^n(c_n x+b_n) = \exp\{n \log F(c_n x+b_n)\} \leq \exp\{-n(1-F(c_n x+b_n))\},$$

hence the convergence of (12) is implied by (10). □

Section 2. In the situation described in the introduction we prove the following statement concerning the rate of growth of $\{Y_n\}$.

Theorem 1. Suppose F is a distribution function with positive derivative F'(x) for all real x. If for some constant c ($0 \leq c < \infty$)

(13) $$\lim_{t\to\infty} \frac{g(t)}{t} = c$$

(with g defined by (2)), then almost surely

(14) $$\begin{cases} \liminf_{n\to\infty} Y_n/b_n = 1 \\ \limsup_{n\to\infty} Y_n/b_n = e^c. \end{cases}$$

Here b_n is defined by $F(b_n) = 1 - 1/n$.

If (13) holds with $c = \infty$, then almost surely $\limsup_{n\to\infty} Y_n/b_n = \infty$.

Remark. For $c = 0$ the theorem has been proved by Geffroy [2].

Proof. We use lemma 1 with $\psi(x) = \log 1/1-F(x)$. Then

$$\frac{\log \psi(t)}{t\ \psi'(t)} = \frac{\{1-F(t)\}\log \log\{1/1-F(t)\}}{t\ F'(t)} = \frac{g(t)}{t} \to c \text{ for } t \to \infty$$

and hence for $x > 0$

$$\lim_{t\to\infty} \log\{\frac{1-F(tx)}{1-F(t)}\} \ .\ \{\log \log 1/1-F(t)\}^{-1} = - \frac{\log x}{c}$$

or equivalently

$$1 - F(tx) = \{1-F(t)\}\ \{\log 1/1-F(t)\}^{c(t)}$$

with

$$\lim_{t\to\infty} c(t) = - \frac{\log x}{c}\ .$$

Substitution of b_n for t gives

$$(15) \qquad 1 - F(b_n x) = \{1-F(b_n)\}\ \{\log 1/1-F(b_n)\}^{r_n} = \frac{(\log n)^{r_n}}{n}$$

with

$$\lim_{n\to\infty} r_n = -\frac{\log x}{c} . \tag{16}$$

First we prove the statement concerning the lim sup for $0 \leq c \leq \infty$. As the righthand side of (16) is less than -1 for $x > e^c$ and larger than -1 for $x < e^c$, we have proved

$$\sum_{n=1}^{\infty} \{1-F(b_n x)\} < \infty \qquad \text{for } x > e^c$$

$$\sum_{n=1}^{\infty} \{1-F(b_n x)\} = \infty \qquad \text{for } x < e^c$$

and by part b) of lemma 4 (with $c_n = b_n$) we have almost surely

$$\limsup_{n\to\infty} Y_n/b_n = e^c.$$

To prove the statement concerning the lim inf for $0 \leq c < \infty$ we verify condition (10) of lemma 4 with $c_n = b_n$. Using (15) we have for $0 < x < 1$

$$\sum_{n=1}^{\infty} \{1-F(b_n x)\} \exp\{-n(1-F(b_n x))\} =$$

$$= \sum_{n=1}^{\infty} n^{-1}(\log n)^{r_n} \exp\{-(\log n)^{r_n}\}.$$

Take $M \geq \frac{-2c}{\log x} + 1$, then

$$\sum_{n=1}^{\infty} \{1-F(b_n x)\} \exp\{-n(1-F(b_n x))\} << \sum_{n=1}^{\infty} n^{-1}(\log n)^{r_n} (\log n)^{-Mr_n} <<$$

$$<< \sum_{n=1}^{\infty} n^{-1}(\log n)^{-3/2} < \infty$$

and we have almost surely

$$\liminf_{n\to\infty} Y_n/b_n \geq 1.$$

By part a) of lemma 4 (with $c_n = b_n$) the proof is complete. □

Remark. In the usual way (see e.g. [2] p. 121) the result can be translated as follows: if $g(x) \to c$ $(0 \leq c \leq \infty)$, then $P\{\limsup_{n\to\infty}(Y_n - b_n) = c\} = 1$; moreover $P\{\liminf_{n\to\infty} Y_n - b_n = 0\} = 1$ for $0 \leq c < \infty$.

For $0 < c < \infty$ this theorem provides exact information concerning the behaviour of Y_n. For $c = 0$ we prove a refined statement.

Theorem 2. Suppose F is a twice differentiable distribution function and $F'(x)$ is positive for all real x. If

(17) $$\lim_{t\to\infty} g'(t) = 0$$

(with g defined by (2)), then almost surely

$$(18)\quad \begin{cases} \liminf_{n\to\infty} \dfrac{Y_n - b_n}{f(b_n)\log\log n} = 0 \\[2ex] \limsup_{n\to\infty} \dfrac{Y_n - b_n}{f(b_n)\log\log n} = 1 \end{cases}$$

(here f is defined by (1) and b_n defined by $F(b_n) = 1-1/n$).

<u>Proof</u>. The proof is similar to that of theorem 1. We use lemma 3 with $\psi(x) = \log 1/1-F(x)$. Then

$$q'(t) = g'(t) \to 0 \qquad\qquad \text{for } t \to \infty$$

and hence

$$\lim_{t\to\infty} \log\{\frac{1-F(t+xg(t))}{1-F(t)}\} \{\log\log 1/1-F(t)\}^{-1} = -x$$

or equivalently

$$1 - F(t+xg(t)) = \{1-F(t)\} \{\log 1/1-F(t)\}^{c(t)}$$

with

$$\lim_{t\to\infty} c(t) = -x.$$

Substitution of b_n for t gives

$$g(b_n) = f(b_n) \log \log 1/1-F(b_n) = f(b_n) \log \log n$$

and

$$(19) \quad 1-F(b_n+xf(b_n)\log \log n) = \{1-F(b_n)\} \{\log 1/1-F(b_n)\}^{r_n} = \frac{(\log n)^{r_n}}{n}$$

with

$$(20) \qquad \lim_{n\to\infty} r_n = -x.$$

We want to apply lemma 4 with $c_n = f(b_n) \log \log n$. By (17) for all real x the sequence $\{b_n+xf(b_n)\log \log n\} = \{b_n+xg(b_n)\}$ is ultimately non-decreasing.

As the righthand member of (20) is less than -1 for x > 1 and larger than -1 for x < 1, we have proved

$$\sum_{n=1}^{\infty} 1 - F(b_n+xf(b_n)\log \log n) < \infty \qquad \text{for } x > 1$$

$$\sum_{n=1}^{\infty} 1 - F(b_n+xf(b_n)\log \log n) = \infty \qquad \text{for } x < 1$$

and by part b) of lemma 4 we have almost surely

$$\limsup_{n\to\infty} \frac{Y_n - b_n}{f(b_n)\log\log n} = 1.$$

By part a) of lemma 4 we have

$$\liminf_{n\to\infty} \frac{Y_n - b_n}{f(b_n)\log\log n} \leq 0.$$

To prove the other statement concerning the lim inf we verify condition (10) of lemma 4. Using (19) and (20) we have for x < 0 with $M \geq -\frac{2}{x} + 1$

$$\sum_{n=1}^{\infty} \{1-F(b_n+xf(b_n)\log\log n)\} \exp\{-n(1-F(b_n+xf(b_n)\log\log n))\}$$

$$= \sum_{n=1}^{\infty} n^{-1}(\log n)^{r_n} \exp\{-(\log n)^{r_n}\} << \sum_{n=1}^{\infty} n^{-1}(\log n)^{r_n(1-M)} <<$$

$$<< \sum_{n=1}^{\infty} n^{-1}(\log n)^{-3/2} < \infty$$

and hence almost surely

$$\liminf_{n\to\infty} \frac{Y_n - b_n}{f(b_n)\log\log n} \geq 0. \qquad \square$$

Remark. Theorem 2 has been stated first by J. Pickands III [5] but the proof seems to contain an error: the distribution function

$$F(x) = 1 - \exp -\{\int_e^x \frac{(\log\log t)^{3/2}}{t} dt\}$$

satisfies the conditions of the theorem but the first relation in the proof does not hold (the limit actually equals infinity).

Remark. Relation (17) implies relation (13) of theorem 1 with c = 0. On the other hand for distribution functions satisfying (13)

$$\lim_{n\to\infty} \frac{f(b_n)\log\log n}{b_n} = c,$$

hence for $0 < c < \infty$ the condition (13) implies

$$\begin{cases} \liminf\limits_{n\to\infty} \dfrac{Y_n - b_n}{f(b_n)\log\log n} = 0 \\ \limsup\limits_{n\to\infty} \dfrac{Y_n - b_n}{f(b_n)\log\log n} = \dfrac{e^c - 1}{c} \end{cases}$$

almost surely.

Examples of distribution functions satisfying theorem 2 are given by Pickands. The distribution functions

$$F(x) = 1 - \exp\{-\int_e^x \frac{(\log\log t)^p}{c.t} dt\}$$

with positive p and c satisfy

$$\lim_{t\to\infty} \frac{g(t)}{t} = \lim_{t\to\infty} g'(t) = \begin{cases} 0 & \text{for } p > 1 \\ c & \text{for } p = 1 \\ \infty & \text{for } p < 1. \end{cases}$$

As all these distribution functions are in the domain of attraction of the double exponential distribution, this answers a question raised by Pickands whether theorem 2 holds for all distribution functions from this domain of attraction.

It is clear that if (18) from theorem 2 holds, then this relation is still true of we replace

$$Y_n = \max(X_1, X_2, \ldots, X_n)$$

by

$$[Y_n] + 1 = \max([X_1]+1, [X_2]+1, \ldots, [X_n]+1)$$

(here [a] is the largest integer not exceeding a). As (18) holds for the exponential distribution with $b_n = \log n$ and $f(b_n) = 1$, this relation is also true for the geometric distribution

$$F(x) = 1 - e^{-[x]} \qquad \text{for } x > 0.$$

Hence the validity of (18) does not imply that F belongs to the domain of attraction of the double exponential distribution.

Section 3. Let us reconsider the condition of theorem 2.

$$g'(t) = \frac{d}{dt}\{\frac{1-F(t)}{F'(t)} \log\log 1/1-F(t)\}$$

$$= \frac{d}{dt}\{\frac{1-F(t)}{F'(t)}\} \log\log 1/1-F(t) + \{\log 1/1-F(t)\}^{-1}$$

$$= f'(t) \,.\, \log\log 1/1-F(t) + o(1) \qquad \text{for } t \to \infty.$$

So $g'(t) \to 0$ for $t \to \infty$ if and only if

(21) $$\lim_{t\to\infty} f'(t) \,.\, \log\log 1/1-F(t) = 0$$

and both imply Von Mises' condition $f'(t) \to 0$ (see [4]) for the domain of attraction of the double exponential distribution. So (21) implies

$$\lim_{n\to\infty} P\{\frac{Y_n - b_n}{f(b_n)} \le x\} = \exp(-e^{-x}).$$

We shall prove a large deviations result related to this weak convergence property under a condition of the type (21).

<u>Theorem 3</u>. Suppose ϕ is a non-decreasing function and $\lim_{x\to\infty} \phi(x) = \infty$. If

$$(22) \qquad \lim_{t\to\infty} f'(t)\, \phi^2(1/1-F(t)) = 0$$

(with f defined by (1)), then

$$(23) \qquad \lim_{n\to\infty} \frac{1-F^n(b_n+x_n f(b_n))}{1-\exp(-e^{-x_n})} = 1$$

for all sequences of positive numbers $\{x_n\}$ with $x_n = O(\phi(n))$ for $n \to \infty$. Here b_n is defined by $F(b_n) = 1-1/n$.

<u>Proof</u>. Obviously (22) implies $f'(t) \to 0$ for $t \to \infty$ and hence by Von Mises' criterion (see [4])

$$\lim_{n\to\infty} F^n(b_n+xf(b_n)) = \exp(-e^{-x})$$

uniformly on each bounded x-interval. Hence (23) holds trivially for each bounded sequence $\{x_n\}$. Next suppose $x_n \to \infty$ for $n \to \infty$. From $-\ln y \sim 1-y$ for $y \uparrow 1$ it follows

$$1 - F^n(b_n+x_n f(b_n)) \sim n\{1-F(b_n+x_n f(b_n))\}$$

and

$$1 - \exp(-e^{-x_n}) \sim e^{-x_n}$$

for $n \to \infty$. So we have to prove

(24) $$\lim_{n\to\infty} ne^{x_n}\{1-F(b_n+x_n f(b_n))\} = 1.$$

by (1) we have

$$\frac{1}{f(t)} = \frac{F'(t)}{1-F(t)}$$

and hence

$$\int_1^x \frac{dt}{f(t)} = -\log\{1-F(x)\} + \log\{1-F(1)\}$$

or equivalently (with $c_0 = 1-F(1)$)

$$1 - F(x) = c_0 \exp\{-\int_1^x \frac{dt}{f(t)}\}.$$

Substitution in (24) gives (as $n = 1/1-F(b_n)$)

$$ne^{x_n}\{1-F(b_n+x_nf(b_n))\} = \exp\{x_n - \int_{b_n}^{b_n+x_nf(b_n)} \frac{ds}{f(s)}\} =$$

$$= \exp\{\int_0^1 -x_n(\frac{f(b_n)}{f(b_n+sx_nf(b_n))} -1)ds\}.$$

As $x_n = O(\phi(n))$, for proving the theorem it is sufficient to show

$$\lim_{n\to\infty} \phi(n)\{\frac{f(b_n)}{f(b_n+xf(b_n)\phi(n))} -1\} = 0 \tag{25}$$

uniformly on any bounded x-interval from $[0,\infty)$. Substitution of t for b_n gives $\phi(n) = \phi(1/1-F(t))$ and (25) becomes

$$\lim_{t\to\infty} \psi(t)\{\frac{f(t)}{f(t+xf(t)\psi(t))} -1\} = 0$$

with $\psi(t) = \phi(1/1-F(t))$.

Using the mean value theorem of differential calculus we get for some $0 \leq \theta(t,x) \leq 1$

$$\psi(t)\{\frac{f(t)}{f(t+xf(t)\psi(t))} - 1\} \tag{26}$$

$$= \frac{\psi(t)}{f(t+xf(t)\psi(t))} (-x)f(t)\psi(t)f'(t+\theta(t,x)xf(t)\psi(t))$$

$$= -x\{\frac{\psi^2(t)}{\psi^2(t+\theta(t,x)xf(t)\psi(t))}\} \{\frac{f(t)}{f(t+xf(t)\psi(t))}\} \ .$$

$$. \{f'(t+\theta(t,x)xf(t)\psi(t))\psi^2(t+\theta(t,x)xf(t)\psi(t))\}.$$

Now we treat the last three factors separately.

As ψ is non-decreasing the first factor is bounded by 1. By assumption the last factor tends to zero uniformly on $[0,\infty)$. As

$$\frac{f(t+xf(t)\psi(t))-f(t)}{f(t)}$$

$$= x\,\frac{\psi(t)}{\psi(t+\theta_1(t,x)xf(t)\psi(t))}\,f'(t+\theta_1(t,x)xf(t)\psi(t))\psi(t+\theta_1(t,x)xf(t)\psi(t))$$

and $\psi(t) \leq \psi^2(t)$ for sufficiently large t, it follows

(27) $$\lim_{t\to\infty} \frac{f(t)}{f(t+xf(t)\psi(t))} = 1$$

uniformly on every bounded x-interval from $[0,\infty)$ and we have proved the theorem. □

Remark. The condition of the theorem cannot be improved essentially: suppose $f'(t)\phi^2(1/1-F(t)) \to c$ with $0 < c < \infty$ and $t\phi'(t) \to 0$, then one can prove

$$\lim_{n\to\infty} \frac{1-F^n(f(b_n)\phi(n)+b_n)}{1-\exp(-e^{-\phi(n)})} = e^{c/2}.$$

As an example we consider the normal distribution. Here

$$f'(t) = te^{t^2/2} \int_t^\infty e^{-s^2/2}\, ds - 1 \sim - t^{-2} \qquad \text{for } t \to \infty.$$

As the inverse function of $1/1-F(t)$ is asymptotically equal to $\sqrt{2 \log s}$, (22) holds if

$$\lim_{t\to\infty} f'(t)\, \phi^2(1/1-F(t)) = \lim_{t\to\infty} - \frac{\phi^2(1/1-F(t))}{t^2} = \lim_{s\to\infty} - \frac{\phi^2(s)}{(2 \log s)} = 0$$

and (23) is true for sequences $\{x_n\}$ with

$$x_n = o(\sqrt{\log n}) \qquad \text{for } n \to \infty.$$

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